Worldline Variational Approximation: A New Approach to the Relativistic Binding Problem

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Abstract

We determine the lowest bound-state pole of the density-density correlator in the scalar Wick-Cutkosky model where two equal-mass constituents interact via the exchange of mesons. This is done by employing the worldline representation of field theory together with a variational approximation as in Feynman's treatment of the polaron. Unlike traditional methods based on the Bethe-Salpeter equation, self-energy and vertex corrections are (approximately) included as are crossed diagrams. Only vacuum-polarization effects of the heavy particles are neglected. The well-known instability of the model due to self-energy effects leads to large qualitative and quantitative changes compared to traditional approaches which neglect them. We determine numerically the critical coupling constant above which no real solutions of the variational equations exist anymore and show that it is smaller than in the one-body case due to an induced instability. The width of the bound state above the critical coupling is estimated analytically.

1. Introduction

Traditionally the relativistic bound-state problem is treated in the framework of the Bethe-Salpeter equation [1] which – although formally exact – has to be approximated in various ways. The most common one is the ladder approximation which nearly has become synonymous with *the* Bethe-Salpeter equation although it has numerous deficiencies. [2] Over the years three-dimensional reductions, spectator approximations, light front methods [3]– to name just a few variants – have been investigated and frequently used. In hadronic physics where the perturbative methods of bound-state QED [4] are of little value there is an urgent need for methods which also work at strong coupling. Lattice Gauge Theory is considered as the prime method to obtain gauge-invariant results from first principles, albeit with enormous numerical effort and problems of its own. The continous progress of lattice calculations not withstanding, considerable progress has also been made in the last years in solving the Dyson-Schwinger equations for Landau-gauge QCD under some simplifications [5] and in describing the low-lying hadrons as bound-state of quarks and gluons. [6] While phenomenologically quite succesful and often going beyond the ladder BSE these calculations still have limitations due to truncations, gauge dependence and the use of model propagators. Due to that the general impression (at least in the high-energy physics community) seems to be that the strong-coupling, relativistic bound-state problem is just so messy that one has to wait for better lattice calculations to determine the hadronic masses from the binding of quarks and gluons. Therefore it may be useful to have a fresh look at this more than 50-year-old problem from the perspective of the particle representation of field theory which has attractive features as demonstrated by Feynman's variational treatment of the polaron. [7] Of course, variational methods have also been used before in field theory [8] and, in particular for the bound-state problem [9], but rather based on fields than on particle trajectories. The huge reduction in degrees of freedom which the the worldline description entails allows to obtain good results with rather crude variational *ansätze*.

2. Variational worldline approximation to the correlator

"In the relativistic approach, bound states and resonances are identified by the occurence of poles in Green functions. A simple extension of the Schrödinger equation is unfortunately not avaliable ... " (p. 481 in Ref. 10). Therefore we look for poles of a special 4-point function, the density-density correlator (or polarization propagator in the language of many-body theory)

$$\Pi(q) := -i \int d^4x \, e^{iq\cdot x} \left\langle 0 \left| \mathcal{T} \left(\hat{\Phi}_2^\dagger(x)\hat{\Phi}_1(x) \, \hat{\Phi}_1^\dagger(0)\hat{\Phi}_2(0) \right) \right| 0 \right\rangle \tag{1}$$

as a function of the external variable q. The correlator (with appropriately modified currents) is precisely the object from which hadronic masses have been estimated in the QCD-sum rule approach. [11] Whereas that method uses a delicate matching between short- and long-distance expansions our aim is to approximate the correlator variationally and to extract the pole position analytically. We do that in the context of the scalar Wick-Cutkosky model [12] where heavy charged particles ("nucleons") interact via the exchange of neutral scalar "mesons" (χ). Its Lagrangian is given by (with $\hbar = c = 1$)

$$\mathcal{L} = \sum_{i=1}^{2} \left[|\partial_\mu \Phi_i|^2 - \left(M_0^2 + 2g\chi \right) |\Phi_i|^2 \right] + \frac{1}{2} (\partial_\mu \chi)^2 - \frac{1}{2}m^2\chi^2 \, . \tag{2}$$

Incidentally, this model (with $m = 0$) was just invented to study the bound-state problem. In the ladder approximation the Bethe-Salpeter equation can be solved exactly (see any field theory textbook which still covers bound-state problems). For simplicity, in the present work we assume that both particles have the same bare mass M_0 but different quantum numbers so that annihilation into mesons is not possible. The coupling constant in Eq. (2) has been written so as to conform with previous work in the one-nucleon sector. [13] Bound states of a nucleon of type 1 and an antinucleon of type 2 will manifest themselves as poles *below* the threshold $q^2 < 4M^2$ where M is the physical mass of the nucleon.

In the quenched approximation where nucleon loops are neglected (so that there is no divergent nucleon-field and coupling-constant renormalization) the correlator can be expressed as a double worldline path integral [14]

$$\Pi(q) = i \int d^4x\, e^{iq \cdot x} \int_0^\infty \frac{dT_1 dT_2}{(2i\kappa_0)^2} \exp\left[-\frac{iM_0^2(T_1 + T_2)}{2\kappa_0} \right]$$
$$\cdot \int \mathcal{D}x_1 \mathcal{D}x_2 \, \exp\left\{ i \sum_{i=1}^2 S_0[x_i] + i S_{\text{int}}[x_1, x_2] \right\} . \tag{3}$$

Here the (4-dimensional) nucleon trajectories have to obey the boundary conditions $x_1(0) = 0$, $x_1(T_1) = x$ and $x_2(0) = x$, $x_2(T_2) = 0$,

$$S_0[x_i] = \int_0^{T_i} dt \left(-\frac{\kappa_0}{2} \dot{x}_i^2(t) \right) \;,\; i = 1, 2 \tag{4}$$

is the standard free action for each particle and

$$S_{\text{int}}[x_1, x_2] = -\frac{g^2}{2\kappa_0^2} \sum_{i,j=1}^2 \int_0^{T_i} dt \int_0^{T_j} dt' \int \frac{d^4p}{(2\pi)^4} \frac{\exp\left[-ip \cdot (x_i(t) - x_j(t')) \right]}{p^2 - m^2 + i0} . \tag{5}$$

the interaction term. The integration over proper times T_1, T_2 arises from the Schwinger representation for each interacting nucleon propagator while $\kappa_0 > 0$ reparametrizes the proper time and can be considered as "mass" of the equivalent quantum mechanical particle. The mesons have been integrated out exactly which leads to a retarded, two-time action with the free meson propagator connecting different points on the worldlines. If Eq. (5) is split into terms with $i = j$ and $i \neq j$ one sees that the former generate the self-energies of each particle while the latter describe the interaction between nucleon and antinucleon by exchange of (any number of) mesons. The vertex corrections come automatically due to different values of the proper times; for example, if one self-energy meson is already "in the air" when another meson is exchanged with the second particle (see Fig. 1). By the same reason all crossed diagrams are also included.

As in Feynman's treatment of the polaron we approximate the highly nonlinear, retarded action by that of a retarded harmonic oscillator with free parameters and retardation functions and apply Jensen's inequality $\langle \exp(-\Delta S) \rangle \geq \exp(-\langle \Delta S \rangle)$ (in real time: stationarity, for details see e.g. Ref. 13). This is best done in Fourier space where each nucleon trajectory (after redefining $t_2 \to T_2 - t_2$) can be written as

$$x_i(t) = x \frac{t}{T_i} + \sum_{k=1}^\infty \frac{\sqrt{2T_i}}{k\pi} a_k^{(i)} \sin\left(\frac{k\pi t}{T_i} \right) \;,\; i = 1, 2 . \tag{6}$$
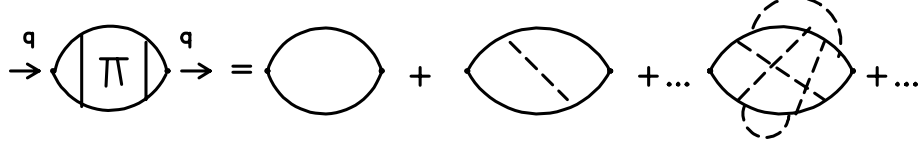
Figure 1: Graphical representation of the correlator $\Pi(q)$ from Eq. (3) and its perturbative expansion. Solid lines refer to nucleons, dashed ones to mesons. Only diagrams where nucleon pairs are created from the vacuum are omitted (quenched approximation).

The functional integration over the trajectories $x_i(t)$ is now replaced by an integration over the Fourier coefficients $a_k^{(i)}$. Including the integral over the final position x into our definition of averages we employ the following quadratic trial action

$$\tilde{S}_t = \tilde{\lambda} q \cdot x - \frac{\kappa_0}{2} \sum_{i=1}^{2} \left[\frac{A_0 x^2}{T_i} + \sum_{k=1}^{\infty} A_k a_k^{(i)\,2} \right] + \kappa_0 \sum_{k=1}^{\infty} B_k a_k^{(1)} \cdot a_k^{(2)} \tag{7}$$

which for $\tilde{\lambda} = A_k = 1$, $B_k = 0$ reduces to the free action. The last term accounts for the direct coupling of the two worldlines. Due to the quadratic trial action the various averages can be evaluated exactly: $\int \exp(i\tilde{S}_t)$ and $\left\langle \tilde{S}_0 - \tilde{S}_t \right\rangle$ give rise to a "kinetic term" Ω_{12} while $\langle S_{\rm int} \rangle$ leads to two distinguished "potentials" corresponding to the self-interaction of each particle (V_{ii}) and the direct interactions between different particles ($V_{ij}, i \neq j$). Similar as in previous applications of the variational worldline approximation a pole develops when the proper time (here the combination $T = (T_1 + T_2)/2$ as shown in more detail in Ref. 15) tends to infinity and only those terms in the exponentials contribute which are proportional to T. Then one obtains *Mano*'s equation [16]

$$M_0^2 = \left(\frac{q}{2} \right)^2 \left(2\lambda - \lambda^2 \right) - \Omega_{12} - 2 \left(V_{11} + V_{12} \right) . \tag{8}$$

Here $\lambda = \tilde{\lambda}/A(0)$ is a modified variational parameter. The kinetic term Ω_{12} acts as a restoring term for the variational principle and depends on the variational parameters A_k which in the limit $T \to \infty$ become a "profile function" $A_k \to A(k\pi/T) \equiv A(E)$. Similarly, $B_k \to B(k\pi/T) \equiv B(E)$. Finally, the interaction terms

$$V_{1j} = \frac{g^2}{2\kappa_0} Z^{j-1} \int_{-\infty}^{+\infty} d\sigma \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 - m^2 + i0} \exp\left\{ \frac{i}{2\kappa_0} \left[p^2 \mu_{1j}^2(\sigma) - \lambda p \cdot q \, \sigma \right] \right\} \tag{9}$$

($j = 1, 2$) depend on the "pseudotimes" $\mu_{1j}^2(\sigma = t - t')$ which basically are Fourier cosine transforms of the inverse profile functions. To keep track of the binding potential V_{12} we have introduced an additional factor Z in front of its proper time integral which will be set to unity at the end of (analytic) calculations. This will allow to distinguish relativistic binding corrections from radiative corrections to the binding energy. A remarkable simplification is achieved by introducing the combinations $A_\pm(E) := A(E) \pm B(E)$. Then the kinetic term becomes $\Omega_{12} = \Omega[A_-] + \Omega[A_+]$ where

$$\Omega[A] = \frac{2\kappa_0}{i\pi} \int_0^\infty dE \left[\ln A(E) + \frac{1}{A(E)} - 1 \right] \tag{10}$$

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is just the usual kinetic term encountered in the self-energy of a single nucleon [13] and the pseudotimes are given by

$$\mu_{1j}^2(\sigma) = \frac{2}{\pi} \int_0^\infty dE \frac{1}{E^2} \left\{ \frac{\delta_{j2}}{A_+(E)} + \left[\frac{1}{A_-(E)} + \frac{(-)^{j+1}}{A_+(E)} \right] \sin^2\left(\frac{E\sigma}{2}\right) \right\} . \tag{11}$$

Closer inspection of the variational equations (see below) reveals that for small proper times $\mu_{11}^2(\sigma) \to \sigma$ exhibits the usual self-energy behaviour but that $\mu_{12}^2(\sigma) \to$ constant which is a new feature due to binding. This implies that only the self-energy part V_{11} develops divergencies which after regularization with a cutoff Λ can be absorbed into a mass renormalization

$$M_0^2 \longrightarrow M_1^2 = M_0^2 - \frac{g^2}{4\pi^2} \ln\left(\frac{\Lambda^2}{m^2}\right) , \tag{12}$$

exactly as in the one-body case. Note that the intermediate mass M_1 is not yet the physical mass M which is obtained from Mano's one-body equation by a finite shift.

3. Numerical results and induced instability

By construction Mano's Eq. (8) is stationary under variation of variational parameters and functions. Performing the variation w.r.t. λ one obtains

$$\lambda = 1 - \frac{4}{q^2} \frac{\partial}{\partial\lambda} (V_{11} + V_{12}) , \tag{13}$$

and variation w.r.t. the profile functions $A_\pm(E)$ gives

$$A_-(E) = 1 + \frac{2i}{\kappa_0} \frac{1}{E^2} \int_0^\infty d\sigma \sum_{j=1}^2 \frac{\delta V_{1j}}{\delta \mu_{1j}^2(\sigma)} \sin^2\left(\frac{E\sigma}{2}\right) \tag{14}$$

$$A_+(E) = 1 + \frac{2i}{\kappa_0} \frac{1}{E^2} \int_0^\infty d\sigma \left[\frac{\delta V_{11}}{\delta \mu_{11}^2(\sigma)} \sin^2\left(\frac{E\sigma}{2}\right) + \frac{\delta V_{12}}{\delta \mu_{12}^2(\sigma)} \cos^2\left(\frac{E\sigma}{2}\right) \right]. \tag{15}$$

The derivatives of the interaction terms are easily worked out and are not given here. Note that the pseudotimes are related to the profile functions by Eq. (11). We have solved these coupled variational equations for $\lambda, A_\pm(E)$ and $\mu_{1j}^2(\sigma)$ numerically in euclidean time [1] with similar iterative methods as described in Ref. 17. Indeed, we have found a pole at $q_0^2 < 4M^2$ for certain ranges of the mass ratio m/M and the standard dimensionless coupling constant

$$\alpha = \frac{g^2}{4\pi M^2} . \tag{16}$$

While more details will be given elsewhere [15], some results for the binding energy $\epsilon = \sqrt{q_0^2} - 2M$ are shown in Fig. 2. The values of the intermediate mass M_1 have been taken from Table III in Ref. 17.

It should be remembered that in the non-relativistic quantum mechanics of two particles interacting via an attractive Yukawa potential (to which our model reduces in the limit $c \to \infty$) binding

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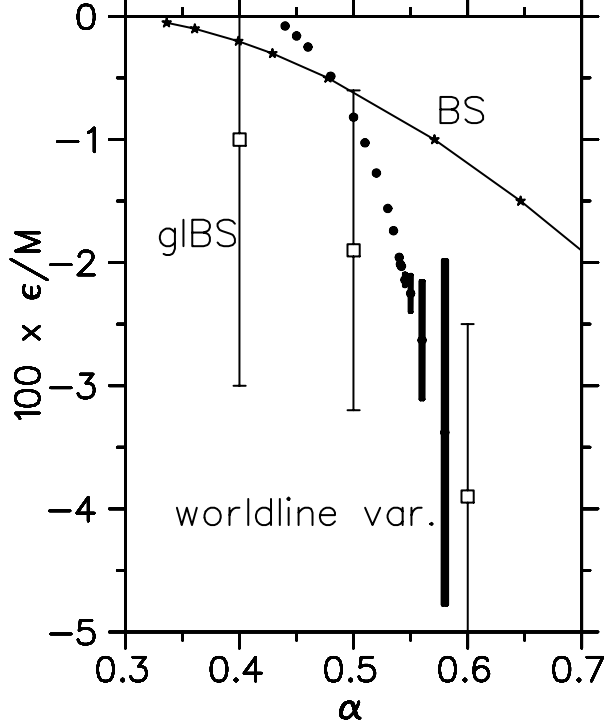[1]This can be simply obtained by setting $\kappa_0 = i$ and reversing the sign of all four-vector products.

Figure 2: The binding energy ϵ/M of the two-body bound state for $M = 0.939$ GeV, $m = 0.14$ GeV as a function of the dimensionless coupling constant defined in Eq. (16). The results from the worldline variational approach including self-energy and vertex corrections are compared with those from Efimov's variational approximation to the ladder Bethe-Salpeter (BS) equation. [18] Also shown are the Monte-Carlo results (with errors) for $m/M = 0.15$ from Ref. 19 in the "generalized ladder approximation" to the Bethe-Salpeter equation (glBS). The thick bars for the worldline results represent the width of the bound state above the critical coupling estimated from Eq. (25).

only occurs for $\delta := \alpha M/m > 1.67981$. [20] The variational approximation with a harmonic oscillator potential requires $\delta > 2.7714$ since it provides only an upper limit for the binding energy. Thus for a finite pion mass a minimal coupling strength is needed to obtain a bound state. Beyond this threshold value we observe much stronger binding when self-energy, vertex corrections and crossed diagrams are taken into account. At first sight this is similar to the results obtained in the "generalized ladder approximation" to the Bethe-Salpeter equation [19] in which all ladder and crossed-ladder diagrams have been included. However, in these Monte-Carlo calculations self-energy and vertex corrections are still neglected which allows binding for arbitrary large coupling constants.

Self-energy corrections to the light-cone Tamm-Dancoff approximation for two nucleons have been considered by Ji [21] who found these corrections to act in a repulsive way. Although approximations in the light-cone and the equal-time formalism are difficult to compare it seems to us that this statement is not valid. This can be best seen in the weak-coupling case for massless mesons where the binding

energy of two equal-mass nucleons takes the form

$$\frac{\epsilon}{M/2} = -b_2 (Z\alpha)^2 \left[1 + r_{21} \frac{\alpha}{\pi} + \dots \right] - b_4 (Z\alpha)^4 \left[1 + \dots \right] - \dots . \tag{17}$$

The exact values of the binding coefficients are $b_2 = 1/2$ (from the non-relativistic Coulomb problem) and $b_4 = 5/32$ (from Todorov's equation for "scalar photons" [22]).

As we include (approximately) self-energy and vertex corrections in our bound-state calculation it is worthwhile to discuss the radiative coefficient r_{21} in more detail: due to vertex corrections the effective coupling constant is enhanced, i.e. r_{21} is positive and *increases* the binding. This enhancement was obtained in Eq. (58) of Ref. 23 and is also the exact one-loop result for a free nucleon because the first-order variational calculation reduces to that in the weak-coupling limit. Alternatively, by standard Feynman-diagram techniques one may calculate the physical amplitude for meson-nucleon scattering at $q = 0$

$$T(q \to 0) = Z_r \Gamma(p, q \to 0) \Big|_{p^2 = M^2} =: g_{\text{eff}} \tag{18}$$

where Z_r is the residue of the 2-point function at the pole and $\Gamma(p, q)$ the truncated meson-nucleon amplitude. From the one-loop diagrams for these quantities one easily obtains

$$g_{\text{eff}} = g \left[1 + \frac{g^2}{4\pi^2} \int_0^1 dx \frac{x^2}{M^2 x^2 + m^2 (1 - x)} \right] \xrightarrow{m=0} g \cdot \left(1 + \frac{\alpha}{\pi} \right) . \tag{19}$$

From Eq. (16) we therefore deduce that $\alpha \to \alpha(1 + 2\alpha/\pi + \dots)$ and that the radiative correction to the coulombic $(Z\alpha)^2$-term is given by $(1 + 4\alpha/\pi + \dots)$, i.e. $r_{21} = 4$. Note that this procedure corresponds to the one-loop determination of the Wilson-coefficient c_1 in the effective non-relativistic field theory of Ref. 24. In such a description explicit antiparticle degrees of freedom and high-energy modes are "integrated out" but their effect is retained in the coefficients of the effective theory. Matching the non-relativistic meson-nucleon scattering amplitude with the one-loop relativistic amplitude for nucleon three-momentum $\mathbf{p} \to 0$ and three-momentum transfer $\mathbf{q} \to 0$ indeed determines $c_1 = g_{\text{eff}}/g$.

What does the worldline variational method predict in the weak-coupling limit for massless mesons ? The answer can be obtained by solving the variational equations (13) - (15) analytically in that limit. As shown elsewhere [15] this gives $b_2^{\text{var}} = 1/\pi$, $r_{21}^{\text{var}} = 7/2$, $b_4^{\text{var}} = 1/\pi^2$.

In view of the fact that both effective field theory and the present approach predict *more* binding (although the latter one with smaller numerical coefficients as expected from a variational calculation) serious doubts remain whether Ji's light-cone calculation is correct. It is unclear to us if this is due to an incomplete nucleon mass renormalization (for a recent discussion see Ref. 25) or because of the restrictive Tamm-Dancoff approximation.

In the worldline variational method an upper limit for the coupling constant comes from the well-known instability of the Wick-Cutkosky model. [26, 27] In the one-body case (i.e. for the two-point function) the polaron variational approximation failed to have real solutions for coupling strengths beyond [2]

$$\alpha_{\text{crit}}^{(1)} = 0.815 . \tag{20}$$

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[2]Similar values have been obtained in truncated Dyson-Schwinger calculations. [28] Below the critical coupling the exact theory would still generate a nonzero, but presumably exponentially small decay width for the particle. Readers uncomfortable with the notion of an unstable ground state (asking "into what does it decay ?") may replace the phrase "decay width" by "inverse lifetime of the metastable state prepared at $t = 0$ ". For recent discussions of the instability see Refs. 27 and 29.

Although a quadratic trial action is incapable of describing tunneling phenomena [30] this is a genuine nonperturbative result which cannot be obtained by an expansion in powers of α. In the present calculations we have found the same phenomenon at *smaller* values of α

$$\alpha_{\text{crit}}^{(2)} \simeq 0.54 \tag{21}$$

implying an *induced* (or catalyzed) instability [31] due to the presence of an additional particle. This can be understood by a particular simple variational *ansatz* first employed in the one-body case [13]: $A_-(E) = 1$, λ free and, additionally

$$A_+(E) = 1 + \frac{\omega^2}{E^2} . \tag{22}$$

Eq. (22) is exactly the euclidean profile function for an harmonic oscillator and accounts for the binding of the particles. Assuming weak binding ($q^2 \simeq 4M^2, \omega \ll M^2$) and massless mesons ($m = 0$) the variational equations for the parameters ω, λ become simple algebraic equations. In particular, the one for the parameter λ is a quartic equation

$$\lambda^4 - \lambda^3 + \frac{\alpha}{2\pi} \lambda^2 + \frac{(Z\alpha)^2}{2\pi} = 0 \tag{23}$$

which generalizes the quadratic equation for the one-body case. Indeed, by setting $Z = 0$ one obtains the former estimate $\alpha_{\text{crit}}^{(1)} \simeq \pi/4 = 0.785$. It is a simple exercise to determine the critical coupling constant from Eq. (23) and one finds

$$\alpha_{\text{crit}}(Z) \simeq \frac{\pi}{8} \frac{(1 + \sqrt{1 + 3z})^3}{(1 + z + \sqrt{1 + 3z})^2} \leq \frac{\pi}{4} , \qquad z := 2\pi Z^2 \tag{24}$$

which for $Z = 1$ gives $\alpha_{\text{crit}}^{(2)} \simeq 0.463$ in fair agreement with the numerical result (21).

It is also possible to determine approximately the width Γ of the bound state for $\alpha > \alpha_{\text{crit}}^{(2)}$ following the very same treatment as in section IV. A of Ref. 17. The idea is to look for a *complex* analytical solution of the variational equations with the simplified ansatz (22). This is only possible if the physical mass acquires an imaginary part, i.e. a width. Close to the critical coupling constant one then finds

$$\Gamma \simeq \frac{2}{3} 2M \left(\frac{\alpha - \alpha_{\text{crit}}^{(2)}}{\alpha_{\text{crit}}^{(2)}} \right)^{3/2} \cdot f_{\text{corr}}(Z) \tag{25}$$

where the correction factor f_{corr} varies between 1 for $Z = 0$ and 1.0885 for $Z = 1$. In view of the rough approximations employed to obtain this result the value of the correction factor is of much less significance than the mass factor $2M$ in Eq. (25) since in the one-body case it simply was M. Thus the instability induced by the presence of the second particle not only shows up in the lower critical coupling constant but also in a much larger width above it.

4. Summary and outlook

We have extended the worldline variational method to the relativistic binding problem of two equal-mass particles in the scalar Wick-Cutkosky model and obtained binding energies which include self-energy, vertex and retardation effects consistently. In this way not only increased binding due to radiative corrections was obtained in the weak-coupling case but also the physics of strong coupling could be addressed. In the latter case, an enhanced instability due to the presence of the second particle was found as a non-perturbative effect in this (admittedly unrealistic) model. From a quantitative viewpoint the major drawback of the present worldline variational method (apart from the quenched approximation) is the relatively poor description of the binding interaction by means of a quadratic trial action. Again, this can be best seen in the weak-coupling expansion (17) where for $m = 0$ not the exact Coulomb result for the leading term is obtained. Different methods developed for the polaron problem could be applied to improve on that: a more general quadratic trial action [29], second-order corrections to Feynman's result [32], Luttinger & Lu's improved variational *ansatz* [33] or Monte-Carlo simulations [19, 34] are possible directions for further work. Extension to gauge theories looks promising since in the one-body sector the worldline variational method does not depend on the covariant gauge parameter [35] in contrast to the Dyson-Schwinger approach. [36] However, already in the present formulation the variational worldline approximation gives novel results for the relativistic bound-state problem by treating in one consistent approximation the two-point (self-energy), three-point (vertex correction), and four-point (binding) effects.

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